Tutorial 8

Bimatrix games

Let A, B be two $m \times n$ matrices. In a two-person game, if A is the payoff matrix for Player I, and B is the payoff matrix for Player II, then we call this game a bimatrix game with bi-matrix (A, B).

1. **Non-cooperative games**

Nash equilibrium

We call a pair of probability vectors $(\boldsymbol{p}, \boldsymbol{q})$ $(\boldsymbol{p} \in \mathcal{P}^m, \boldsymbol{q} \in \mathcal{P}^n)$ a Nash equilibrium for (A, B) if

(i) $\boldsymbol{p}B\boldsymbol{y}^T \leq \boldsymbol{p}B\boldsymbol{q}^T$, for any $\boldsymbol{y} \in \mathcal{P}^n$.

(ii) $\boldsymbol{x}A\boldsymbol{q}^T \leq \boldsymbol{p}A\boldsymbol{q}^T$, for any $\boldsymbol{x} \in \mathcal{P}^m$.

Theorem 1 (Nash Theorem)**.** *Every bimatrix game has at least one Nash equilibrium.*

Solve a non-cooperative game: find all Nash equilibria and the corresponding payoff pairs.

The case that A, B are 2×2 matrices

In this case, there is a simple method to find all Nash equilibria: for $x, y \in [0, 1]$, let

$$\pi(x, y) = \begin{pmatrix} x & 1-x \end{pmatrix} A \begin{pmatrix} y \\ 1-y \end{pmatrix}, \rho(x, y) = \begin{pmatrix} x & 1-x \end{pmatrix} B \begin{pmatrix} y \\ 1-y \end{pmatrix}$$

be the payoff functions of Player I and Player II respectively. Find two sets

$$P = \{(x, y) : \pi(x, y) \text{ attains its maximum at } x \text{ for fixed } y\},$$

$$Q = \{(x, y) : \rho(x, y) \text{ attains its maximum at } y \text{ for fixed } x\}.$$

Then the set of all Nash equilibria is given by

$$\{(\boldsymbol{p}, \boldsymbol{q}) : \boldsymbol{p} = (x, 1-x), \boldsymbol{q} = (y, 1-y), (x, y) \in P \cap Q\}.$$

2. **Cooperative games**

Nash bargaining model

We call an $m \times n$ matrix $P = (p_{ij})$ a probability matrix if $p_{ij} \geq 0$ and $\sum_{i,j} p_{ij} = 1$. In this case, we write $P \in \mathcal{P}^{m \times n}$.

In a cooperative game, each $P \in \mathcal{P}^{m \times n}$ gives a **joint strategy**, and we denote the corresponding payoff to Player I and Player II by

$$u(P) = \sum_{i,j} p_{ij} a_{ij}, \ v(P) = \sum_{i,j} p_{ij} b_{ij}.$$

Cooperative region:

$$\begin{aligned}\mathcal{R} &:= \operatorname{conv}(\{(a_{ij}, b_{ij}) : 1 \leq i \leq m, 1 \leq j \leq n\}) \\ &= \left\{ \sum_{ij} p_{ij}(a_{ij}, b_{ij}) : P = (p_{ij}) \in \mathcal{P}^{m \times n} \right\}.\end{aligned}$$

Status quo point: Usually, we let this point be

$$(\mu, \nu) = (v_A, v_{B^T}).$$

Pareto optimal point: a point $(u, v) \in \mathcal{R}$ is said to pareto optimal if

$$u' \geq u, v' \geq v \Rightarrow u' = u, v' = v.$$

Bargaining set: define the bargaining set to be

$$\{\text{pareto optimal points}\} \cap \{(u,v) \in \mathcal{R} : u \geq \mu, v \geq \nu\}.$$

Bargaining function: let $U = \{(u,v) : u > \mu, v > \nu\}$. Define the bargaining function by

$$g(u,v) = \begin{cases} (u-\mu)(v-\nu) & \text{if } U \neq \emptyset, \\ u+v & \text{otherwise.} \end{cases}$$

Arbitration pair: define the arbitration pair to be the unique point (α, β) in $\mathcal{R}$, such that

$$g(\alpha, \beta) = \max\{g(u,v) : (u,v) \in \text{bargaining set}\}.$$

Exercise 1. *Consider a two-person game with bimatrix*

$$(A,B) = \begin{pmatrix} (2,1) & (4,3) \\ (6,2) & (3,1) \end{pmatrix}.$$

(i) Find v_A, v_{B^T}.

(ii) Find all Nash equlibria.

(iii) Find and sketch the bargaining set. Find the arbitration pair.

Solution. (i) For $x \in [0,1]$, we have

$$(x, 1-x)A = (x, 1-x)\begin{pmatrix} 2 & 4 \\ 6 & 3 \end{pmatrix} = (6-4x, 3+x).$$

Let $6-4x=3+x$, we have $x=\frac{3}{5}$ and $v_A=\frac{18}{5}$. Similarly, we have

$$(x,1-x)B^T=(x,1-x)\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}=(3-2x,1+x).$$

Let $3-2x=1+x$, we have $x=\frac{2}{3}$ and $v_{B^T}=\frac{5}{3}$.

(ii) For $x,y\in[0,1]$, let

$$\pi(x,y)=\begin{pmatrix}x & 1-x\end{pmatrix}A\begin{pmatrix}y\\ 1-y\end{pmatrix},\rho(x,y)=\begin{pmatrix}x & 1-x\end{pmatrix}B\begin{pmatrix}y\\ 1-y\end{pmatrix}.$$

We need to find

$$P=\{(x,y):\pi(x,y)\text{ attains its maximum at }x\text{ for fixed }y\},$$

$$Q=\{(x,y):\rho(x,y)\text{ attains its maximum at }y\text{ for fixed }x\}.$$

To find the set P, consider

$$A\begin{pmatrix}y\\ 1-y\end{pmatrix}=\begin{pmatrix}2 & 4\\ 6 & 3\end{pmatrix}\begin{pmatrix}y\\ 1-y\end{pmatrix}=\begin{pmatrix}4-2y\\ 3+3y\end{pmatrix}.$$

Then we have $4-2y=3+3y$ if $y=\frac{1}{5}$, $4-2y>3+3y$ if $0\le y<\frac{1}{5}$ and $4-2y<3+3y$ if $\frac{1}{5}<y\le y$. Hence

$$P=\left\{(x,\frac{1}{5}):0\le x\le 1\right\}\bigcup\left\{(1,y):0\le y<\frac{1}{5}\right\}\bigcup\left\{(0,y):\frac{1}{5}<y\le 1\right\}.$$

To find the set Q, consider

$$(x,1-x)B=(x,1-x)\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix}=(2-x,2x+1).$$

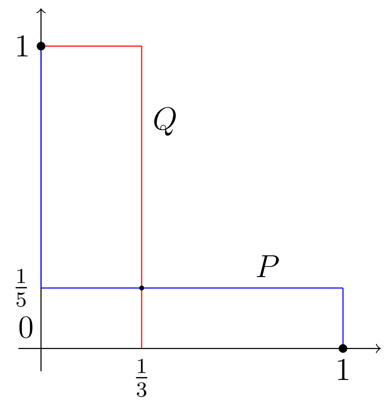

Figure 1

We have $2 - x = 2x + 1$ if $x = \frac{1}{3}$, $2 - x > 2x + 1$ if $0 \leq x < \frac{1}{3}$ and $2 - x < 2x + 1$ if $\frac{1}{3} < x \leq 1$. Hence

$$Q = \left\{(\frac{1}{3}, y) : 0 \leq y \leq 1\right\} \bigcup \left\{(x, 1) : 0 \leq x < \frac{1}{3}\right\} \bigcup \left\{(x, 0) : \frac{1}{3} < x \leq 1\right\}.$$

Draw the graph of P and Q as in Figure 1. Hence we have

$$P \cap Q = \left\{(0, 1), (\frac{1}{3}, \frac{1}{5}), (1, 0)\right\}.$$

For $\boldsymbol{p} = (0, 1), \boldsymbol{q} = (1, 0)$,

$$\pi(\boldsymbol{p}, \boldsymbol{q}) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 6, \ \rho(\boldsymbol{p}, \boldsymbol{q}) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2.$$

Similarly, we have for $\boldsymbol{p} = (1, 0), q = (0, 1)$, $\pi(\boldsymbol{p}, \boldsymbol{q}) = 4$, $\rho(\boldsymbol{p}, \boldsymbol{q}) = 3$ and for $\boldsymbol{p} = (\frac{1}{3}, \frac{2}{3}), \boldsymbol{q} = (\frac{1}{5}, \frac{4}{5})$, $\pi(\boldsymbol{p}, \boldsymbol{q}) = \frac{18}{5}$, $\rho(\boldsymbol{p}, \boldsymbol{q}) = \frac{5}{3}$. We may list the Nash equilibria and the corresponding payoff pairs in the following table.

$\boldsymbol{p}$	$\boldsymbol{q}$	(π, ρ)
$(0, 1)$	$(1, 0)$	$(6, 2)$
$(1, 0)$	$(0, 1)$	$(4, 3)$
$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{5}, \frac{4}{5})$	$(\frac{18}{5}, \frac{5}{3})$

3. Cooperative games

Exercise 2. *Consider bimatrix game*

$$(A, B) = \begin{pmatrix} (a, 2) & (3, 0) \\ (2, 0) & (2, 2) \end{pmatrix},$$

where $a > 2$ is arbitrary. Find the maximin values of the two players and the arbitrary pair as functions of a.